Shape optimization for the eigenvalues of a biharmonic Steklov problem

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joint work with Davide Buoso
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Università degli Studi di Padova

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, $\tau > 0$ a fixed constant.

$$\begin{cases} \Delta^2 u - \tau \Delta u = 0, & \text{in } \Omega\,, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega\,, \\ \tau \frac{\partial u}{\partial \nu} - \mathrm{div}_{\partial\Omega}\left(D^2 u.\nu\right) - \frac{\partial \Delta u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega\,, \end{cases}$$

The biharmonic Steklov problem

$$\int_{\Omega} D^2u : D^2\phi + \tau \nabla u \cdot \nabla \phi dx = \lambda \int_{\partial\Omega} u\phi d\sigma \,, \quad \forall \phi \in H^2(\Omega),$$

where $D^2u : D^2\phi = \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j}$

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$\max_\Omega \lambda_j[\Omega]$? $\min_\Omega \lambda_j[\Omega]$? Critical points ?

among sets Ω with a fixed volume $|\Omega|$

Other Steklov-type problems

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$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega\,, \\ u = 0, & \text{on } \partial\Omega\,, \\ \Delta u = \lambda \frac{\partial u}{\partial \nu}, & \text{on } \partial\Omega\,, \end{cases}$$

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Bucur, Ferrero, Gazzola, "*On the first eigenvalue of a fourth order Steklov problem*", Calc. Var. Partial Differential Equations, 35.

Steklov problem for the Laplacian

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The ball is a maximizer for $\lambda_2[\Omega]$ among Ω with a fixed volume (Weinstock, Brock).

Neumann vs Steklov, second order

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$$\rho_\varepsilon = \begin{cases} \varepsilon, & \text{in } \Omega \setminus \bar{\omega}_\varepsilon\,, \\ C_\varepsilon, & \text{in } \omega_\varepsilon\,, \end{cases}$$

$\omega_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$ and $\int_\Omega \rho_\varepsilon = M$ for all $\varepsilon \in]0, \varepsilon_0[$.

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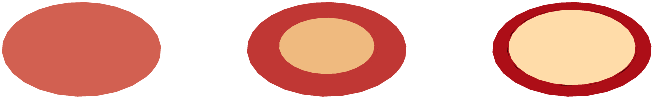

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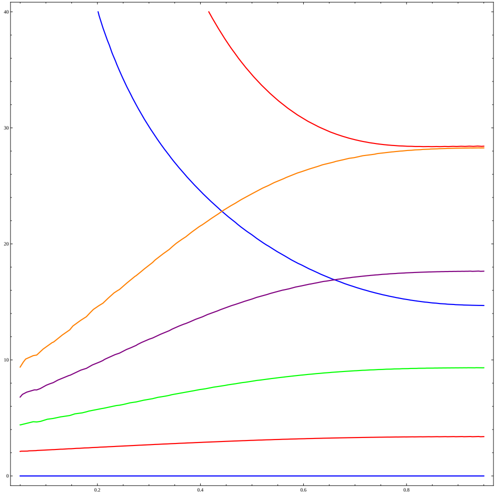

Figure: N=2, M=π

The biharmonic Steklov problem

Strategy:

- Biharmonic Neumann problem with mass density ρ_ε

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Let Ω a bounded domain in $\mathbb{R}^N$. Set

$$\Phi(\Omega) = \left\{\phi \in \left(C^2(\Omega)\right)^N, \text{ injective } : \inf_{\Omega} |\det D\phi| > 0\right\}$$

Symmetric functions of the eigenvalues

Let Ω a bounded domain in $\mathbb{R}^N$. Set

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Theorem (Buoso-P. 2014)

Let Ω be a bounded domain of $\mathbb{R}^N$ of class C^1. Let F be a finite non-empty subset of $\mathbb{N} \setminus \{0\}$. Let

$$\mathcal{A}_\Omega[F] = \left\{ \phi \in \Phi(\Omega) \ : \ \lambda_l[\phi] \notin \left\{ \lambda_j[\phi] : j \in F \right\} \forall l \in \mathbb{N} \setminus (F \cup \{0\}) \right\}$$

Then the set $\mathcal{A}_\Omega$ is open in $\Phi(\Omega)$ and the map $\Lambda_{F,s}$ from $\mathcal{A}_\Omega$ to $\mathbb{R}$ defined by

$$\Lambda_{F,s}[\phi] = \sum_{j_1 < \cdots < j_s \in F} \lambda_{j_1}[\phi] \cdots \lambda_{j_s}[\phi]$$

for $s \in \{1, ..., |F|\}$ is real analytic.

Isovolumetric perturbations

Theorem (Buoso-P. 2014)

Let Ω be a bounded domain in $\mathbb{R}^N$. Let F a finite non-empty subset of $\mathbb{N} \setminus \{0\}$. Let $\tilde{\phi} \in \mathcal{A}_\Omega[F]$ be such that all the eigenvalues with indexes in F have a commond value λ_F and moreover that $\partial\tilde{\phi}(\Omega) \in C^4$. Let $v_1, ..., v_{|F|}$ be a hortonormal basis of the eigenspace associated with the eigenvalue $\lambda_F[\tilde{\phi}]$. Then

$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] = -\lambda_F^s[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}\int_{\partial\tilde{\phi}(\Omega)}\Big(\lambda_F K v_l^2 + \lambda_F\frac{\partial(v_l^2)}{\partial\nu} - \tau|\nabla v_l|^2 - |D^2 v_l|^2\Big)\mu\cdot\nu d\sigma, \quad (1.3)$$

for all $\psi \in (C^2(\Omega))^N$, where $\mu = \psi \circ \phi^{(-1)}$, and K denotes the mean curvature on $\partial\tilde{\phi}(\Omega)$.

Isovolumetric perturbations

$$\mathcal{V}(\phi) = \int_{\phi(\Omega)} dy = \int_{\Omega} |\mathrm{det} D\phi| dx$$

Fix $\mathcal{V}_0 \in]0, +\infty[$

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The function $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ if and only if

$$\sum_{l=1}^{|F|} \left(\lambda_F[\tilde{\phi}] \left(K v_l^2 + \frac{\partial v_l^2}{\partial \nu} \right) - \tau |\nabla v_l|^2 - |D^2 v_l|^2 \right) = c, \text{ a.e. on } \partial\tilde{\phi}(\Omega).$$

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Theorem (Buoso-P. 2014)

Let Ω be a domain of $\mathbb{R}^N$. Let $\tilde{\phi} \in \Phi(\Omega)$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of the problem in $\tilde{\phi}(\Omega)$, and let F be the set of $j \in \mathbb{N} \setminus \{0\}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$. Then $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ on $V(\mathcal{V}(\tilde{\phi}))$, for all $s = 1, \ldots, |F|$.

The fundamental tone

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Among all bounded domains of class C^1 with fixed volume, the ball maximizes the first non-negative eigenvalue, that is $\lambda_2[\Omega] \leq \lambda_2[\Omega^*]$, where Ω^* is the ball with the same volume as Ω.

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$$u(r, \theta_1, ..., \theta_{N-1}) = R_l(r) Y_l(\theta_1, ..., \theta_{N-1})$$

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Example: $g(0, N, \tau) = 0$, $g(1, N, \tau) = \tau$. Which $l \in \mathbb{N}$ gives the fundamental tone?

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Strategy: use the eigenfunctions of the unit ball as test functions in a variational characterization of $\lambda_2[\Omega]$

The fundamental tone

Lemma (Hile-Xu 1993)

Let Ω be a bounded domain of class C^1 in $\mathbb{R}^N$. Then

$$\sum_{l=2}^{N+1} \frac{1}{\lambda_l(\Omega)} = \max\left\{\sum_{l=2}^{N+1} \int_{\partial\Omega} v_l^2 d\sigma\right\},$$

where $\{v_l\}_{l=2}^{N+1}$ is a family in $H^2(\Omega)$ satisfying $\int_\Omega D^2 v_i : D^2 v_j + \tau \nabla v_i \cdot \nabla v_j dx = \delta_{ij}$ and $\int_{\partial\Omega} v_l d\sigma = 0$ for all $l = 2, ..., N+1$.

Lemma (Betta-Brock-Mercaldo-Posteraro 1999)

Let Ω be an open set in $\mathbb{R}^N$ and f be a continuous, non-negative, non-decreasing function defined on $[0, +\infty)$. Let us assume that the function

$$t \mapsto \big(f(t^{1/N}) - f(0)\big)t^{1-(1/N)}$$

is convex. Then

$$\int_{\partial\Omega} f(|x|)d\sigma \geq \int_{\partial\Omega^*} f(|x|)d\sigma,$$

where Ω^* is the ball centered at zero with the same volume as Ω.

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$$\sum_{l=2}^{N+1} \frac{1}{\lambda_l[\Omega]} \geq \frac{1}{\tau|\Omega|}\int_{\partial\Omega} |x|^2 d\sigma \geq \frac{1}{\tau|\Omega|}\int_{\partial B} |x|^2 d\sigma = \frac{N}{\tau} = \sum_{l=2}^{N+1} \frac{1}{\lambda_l[B]}.$$

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Remark: for general values of $|\Omega|$ just observe

$$\lambda[\tau, \Omega] = s^4 \lambda[s^{-2}\tau, s\Omega]$$

Further directions: the case $\tau = 0$

Let $\tau = 0$ and Ω be a bounded domain of class C^1

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ -\mathrm{div}_{\partial\Omega}\big(D^2 u.\nu\big) - \frac{\partial \Delta u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega, \end{cases}$$

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The kernel is $\{1, x_1, ..., x_N\}$

Further directions: the case $\tau = 0$

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- identify the fundamental tone of the unit ball

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- construct trial functions of the form $R(r) Y_2(\theta_1, ...\theta_{N+1})$

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What we (still) cannot do:

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Trial functions work with radial domains. For small dimensions we have isoperimetric inequality

Theorem (Buoso-P. 2014)

Among all bounded radial domains Ω with a fixed volume in $\mathbb{R}^N$, $N \leq 4$, the ball maximizes the first non-zero eigenvalue, that is

$$\lambda_{N+2}[\Omega] \leq \lambda_{N+2}[\Omega^*],$$

where Ω^* is the ball with the same volume of Ω.

Further directions: the case $\tau = 0$

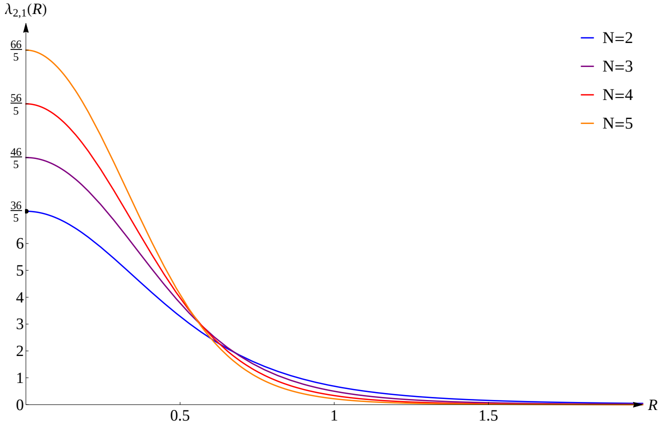

Figure: N=2,3,4,5

Further directions: Neumann problem, Poly-harmonic operators,...

Neumann problem for the biharmonic operator

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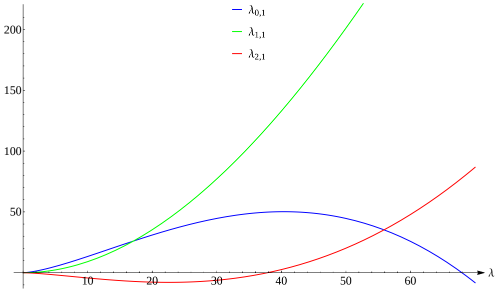

Figure: N=2

Further directions: Neumann problem, Poly-harmonic operators,...

- Neumann problem for $(-\Delta)^m$

$$
\begin{cases}
(-\Delta)^m u = \lambda u, & \text{in } \Omega, \\
N_1 u = N_2 u = \cdots = N_m u = 0, & \text{on } \partial\Omega,
\end{cases}
$$

$N_i u$ are the m natural boundary conditions, ordered according their order: N_1 is an operator of order m, N_2 is of order $m+1$,..., N_m is of order $2m-1$.

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- Neumann problem for $(-\Delta)^m$

$$\begin{cases} (-\Delta)^m u = \lambda u, & \text{in } \Omega, \\ N_1 u = N_2 u = \cdots = N_m u = 0, & \text{on } \partial\Omega, \end{cases}$$

$N_i u$ are the m natural boundary conditions, ordered according their order: N_1 is an operator of order m, N_2 is of order $m+1$,..., N_m is of order $2m-1$.

- Steklov problem for $(-\Delta)^m$

$$\begin{cases} \Delta^m u = 0, & \text{in } \Omega, \\ N_1 u = N_2 u = \cdots = N_{m-1} u = 0, & \text{on } \partial\Omega, \\ N_m u = \lambda u, & \text{on } \partial\Omega, \end{cases}$$

with the same N_i

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THANK YOU

Further directions: mass concentration

- Behavior of $\lambda_j(\varepsilon)$ for mass concentration problem for the biharmonic operator

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- Behavior of $\lambda_j(\varepsilon)$ for mass concentration problem for the biharmonic operator

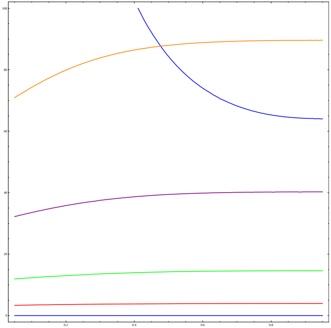

Figure: N=2, M=π, $\tau = 5$

Further directions: mass concentration

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- Behavior of $\lambda_j(\varepsilon)$ for mass concentration problem for the biharmonic operator

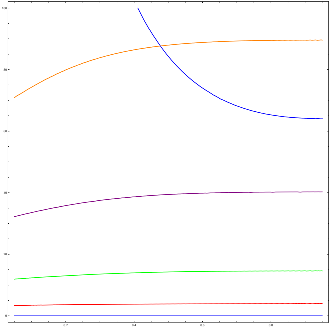

Figure: N=2, M=π, $\tau = 5$

- On the ball? On arbitrary Ω (also in the second order case)?