On the eigenvalues of a fourth-order Steklov problem

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joint work with Davide Buoso

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The Biharmonic Steklov problem

Let Ω be a bounded domain in $\mathbb{R}^N$, $\tau > 0$ a fixed constant.

$$\begin{cases} \Delta^2 u - \tau \Delta u = 0, & \text{in } \Omega\,, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega\,, \\ \tau \frac{\partial u}{\partial \nu} - \mathrm{div}_{\partial\Omega}\left(D^2 u.\nu\right) - \frac{\partial \Delta u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega\,, \end{cases}$$

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$$0 = \lambda_1[\Omega] < \lambda_2[\Omega] \le \cdots \le \lambda_j[\Omega] \le \cdots$$

$$\Omega \mapsto \lambda_j[\Omega]\,, \quad \Omega \mapsto \lambda_2[\Omega]$$

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$\max_\Omega \lambda_j[\Omega]$? $\min_\Omega \lambda_j[\Omega]$? Critical points ?

among sets Ω with a fixed volume $|\Omega|$

Let Ω be a bounded domain of class C^1 in $\mathbb{R}^N$

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega\,, \\ u = 0, & \text{on } \partial\Omega\,, \\ \Delta u = \lambda \frac{\partial u}{\partial \nu}, & \text{on } \partial\Omega\,, \end{cases}$$

Bucur, Ferrero, Gazzola, "*On the first eigenvalue of a fourth order Steklov problem*", Calc. Var. Partial Differential Equations, 35.

Steklov problem for the Laplacian

$$\begin{cases} \Delta u = 0, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega\,, \end{cases}$$

$$0 = \lambda_1[\Omega] < \lambda_2[\Omega] \leq \cdots \leq \lambda_j[\Omega] \leq \cdots$$

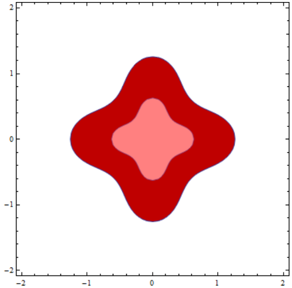
The ball is a maximizer for $\lambda_2[\Omega]$ among Ω with a fixed volume (Weinstock, Brock).

$$\begin{cases} -\Delta u = \lambda(\varepsilon)\rho_\varepsilon u, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\,, \end{cases}$$

where

$$\rho_\varepsilon = \begin{cases} \varepsilon, & \text{if } \operatorname{dist}(x,\partial\Omega) > \varepsilon\,, \\ C(\varepsilon), & \text{if } \operatorname{dist}(x,\partial\Omega) < \varepsilon\,. \end{cases}$$

$C(\varepsilon)$ is s.t. $\int_\Omega \rho_\varepsilon = M$ for all $\varepsilon > 0$.



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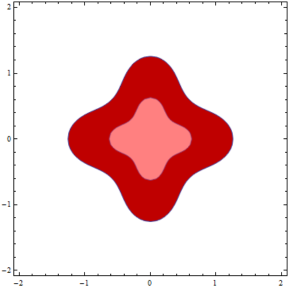

For all $j \in \mathbb{N}$, $\lambda_j(\varepsilon) \to \lambda_j$ as $\varepsilon \to 0$

The Biharmonic Steklov problem

Strategy:

- Biharmonic Neumann problem with mass density ρ_ε

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The ball is a maximizer for $\lambda_2[\Omega]$ among Ω with a fixed volume, when $\rho_\varepsilon \equiv \mathrm{const}$ (Chasman 2011).

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Let Ω a bounded domain in $\mathbb{R}^N$. Set

$$\Phi(\Omega) = \left\{\phi \in \left(C^2(\Omega)\right)^N, \text{ injective } : \inf_{\Omega} |\text{det} D\phi| > 0\right\}$$

Symmetric functions of the eigenvalues

Let Ω a bounded domain in $\mathbb{R}^N$. Set

$$\Phi(\Omega) = \left\{ \phi \in \left(C^2(\Omega)\right)^N, \text{ injective } : \inf_{\Omega} |\text{det} D\phi| > 0 \right\}$$

Theorem (Buoso-P. 2014)

Let Ω be a bounded domain of $\mathbb{R}^N$ of class C^1. Let F be a finite non-empty subset of $\mathbb{N} \setminus \{0\}$. Let

$$\mathcal{A}_{\Omega}[F] = \left\{ \phi \in \Phi(\Omega) \ : \ \lambda_l[\phi] \notin \left\{ \lambda_j[\phi] : j \in F \right\} \forall l \in \mathbb{N} \setminus (F \cup \{0\}) \right\}$$

Then the set $\mathcal{A}_{\Omega}$ is open in $\Phi(\Omega)$ and the map $\Lambda_{F,s}$ from $\mathcal{A}_{\Omega}$ to $\mathbb{R}$ defined by

$$\Lambda_{F,s}[\phi] = \sum_{j_1 < \cdots < j_s \in F} \lambda_{j_1}[\phi] \cdots \lambda_{j_s}[\phi]$$

for $s \in \{1, ..., |F|\}$ is real analytic.

Isovolumetric perturbations

Theorem (Buoso-P. 2014)

Let Ω be a bounded domain in $\mathbb{R}^N$. Let F a finite non-empty subset of $\mathbb{N} \setminus \{0\}$. Let $\tilde{\phi} \in \mathcal{A}_\Omega[F]$ be such that all the eigenvalues with indexes in F have a commond value λ_F and moreover that $\partial\tilde{\phi}(\Omega) \in C^4$. Let $v_1, ..., v_{|F|}$ be a hortonormal basis of the eigenspace associated with the eigenvalue $\lambda_F[\tilde{\phi}]$. Then

$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] = -\lambda_F^s[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l=1}^{|F|}\int_{\partial\tilde{\phi}(\Omega)}\Big(\lambda_F K v_l^2 + \lambda_F\frac{\partial(v_l^2)}{\partial\nu} - \tau|\nabla v_l|^2 - |D^2 v_l|^2\Big)\mu\cdot\nu d\sigma, \quad (1.3)$$

for all $\psi \in (C^2(\Omega))^N$, where $\mu = \psi \circ \phi^{(-1)}$, and K denotes the mean curvature on $\partial\tilde{\phi}(\Omega)$.

Isovolumetric perturbations

$$\mathcal{V}(\phi) = \int_{\phi(\Omega)} dy = \int_{\Omega} |\mathrm{det} \mathrm{D}\phi| dx$$

Fix $\mathcal{V}_0 \in]0, +\infty[$

$$V(\mathcal{V}_0) = \{\phi \in \Phi[\Omega] \, : \, \mathcal{V}(\phi) = \mathcal{V}_0\}$$

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Theorem (Buoso-P. 2014)

Let Ω be a domain of $\mathbb{R}^N$. Let $\tilde{\phi} \in \Phi(\Omega)$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of the problem in $\tilde{\phi}(\Omega)$, and let F be the set of $j \in \mathbb{N} \setminus \{0\}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$. Then $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ on $V(\mathcal{V}(\tilde{\phi}))$, for all $s = 1, \ldots, |F|$.

Balls are critical for the symmetric functions of the eigenvalues under isovolumetric perturbations

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Could we say more on the fundamental tone λ_2 ?

The fundamental tone

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Theorem (Buoso-P. 2014)

Among all bounded domains of class C^1 with fixed volume, the ball maximizes the first non-negative eigenvalue, that is $\lambda_2[\Omega] \leq \lambda_2[\Omega^*]$, where Ω^* is the ball with the same volume as Ω.

The fundamental tone

Consider $B = B(0,1) \subset \mathbb{R}^N$. All the eigenfunctions of the Steklov problem are of the form

$$u(r, \theta_1, ..., \theta_{N-1}) = R_l(r) Y_l(\theta_1, ..., \theta_{N-1})$$

where

$$R_l(r) = \alpha_l r^l + \beta_l i_l(\sqrt{\tau} r).$$

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The corresponding eigenvalues are given by an explicit formula (rather long)

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$$\lambda_2[B] = g(1, N, \tau) = \tau$$

The fundamental tone

Strategy: use the eigenfunctions of the unit ball as test functions in a variational characterization of $\lambda_2[\Omega]$

The fundamental tone

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Lemma (Hile-Xu 1993)

Let Ω be a bounded domain of class C^1 in $\mathbb{R}^N$. Then

$$\sum_{l=2}^{N+1} \frac{1}{\lambda_l(\Omega)} = \max\left\{\sum_{l=2}^{N+1} \int_{\partial\Omega} v_l^2 d\sigma\right\},$$

where $\{v_l\}_{l=2}^{N+1}$ is a family in $H^2(\Omega)$ satisfying $\int_\Omega D^2 v_i : D^2 v_j + \tau \nabla v_i \cdot \nabla v_j dx = \delta_{ij}$ and $\int_{\partial\Omega} v_l d\sigma = 0$ for all $l = 2, ..., N+1$.

Lemma (Betta-Brock-Mercaldo-Posteraro 1999)

Let Ω be an open set in $\mathbb{R}^N$ and f be a continuous, non-negative, non-decreasing function defined on $[0, +\infty)$. Let us assume that the function $t \mapsto \left(f(t^{1/N}) - f(0)\right)t^{1-(1/N)}$ is convex. Then

$$\int_{\partial\Omega} f(|x|) d\sigma \geq \int_{\partial\Omega^*} f(|x|) d\sigma,$$

where Ω^* is the ball centered at zero with the same volume as Ω.

The fundamental tone

Take Ω of class C^1 with $|\Omega| = |B|$ and perform the translation

$$x_i = y_i - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} y_i d\sigma$$

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Use test functions $v_l = (\tau|\Omega|)^{-\frac{1}{2}} x_l$ in the variational formula and use the isoperimetric inequality

$$\sum_{l=2}^{N+1} \frac{1}{\lambda_l[\Omega]} \geq \frac{1}{\tau|\Omega|}\int_{\partial\Omega} |x|^2 d\sigma \geq \frac{1}{\tau|\Omega|}\int_{\partial B} |x|^2 d\sigma = \frac{N}{\tau} = \sum_{l=2}^{N+1} \frac{1}{\lambda_l[B]}.$$

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Remark: for general values of $|\Omega|$ just observe

$$\lambda[\tau, \Omega] = s^4 \lambda[s^{-2}\tau, s\Omega]$$

Further directions: the case $\tau = 0$

Let $\tau = 0$ and Ω be a bounded domain of class C^1

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ -\mathrm{div}_{\partial\Omega}\big(D^2 u.\nu\big) - \frac{\partial \Delta u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega, \end{cases}$$

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The kernel is $\{1, x_1, ..., x_N\}$

Further directions: the case $\tau = 0$

What we can do:

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$$u(r, \theta_1, ..., \theta_{N-1}) = (6r^2 - r^4)Y_2(\theta_1, ..., \theta_{N-1})$$

- construct trial functions of the form $R(r)Y_2(\theta_1, ...\theta_{N+1})$

Further directions: the case $\tau = 0$

What we (still) cannot do:

- test these trial functions on any Ω of class C^1

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Trial functions work with radial domains. For small dimensions we have isoperimetric inequality

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- test these trial functions on any Ω of class C^1
- find good estimates for the sum of the reciprocals in the case the test is possible

Trial functions work with radial domains. For small dimensions we have isoperimetric inequality

Theorem (Buoso-P. 2014)

Among all bounded radial domains Ω with a fixed volume in $\mathbb{R}^N$, $N \leq 4$, the ball maximizes the first non-zero eigenvalue, that is

$$\lambda_{N+2}[\Omega] \leq \lambda_{N+2}[\Omega^*],$$

where Ω^* is the ball with the same volume of Ω.

Further directions: the case $\tau = 0$

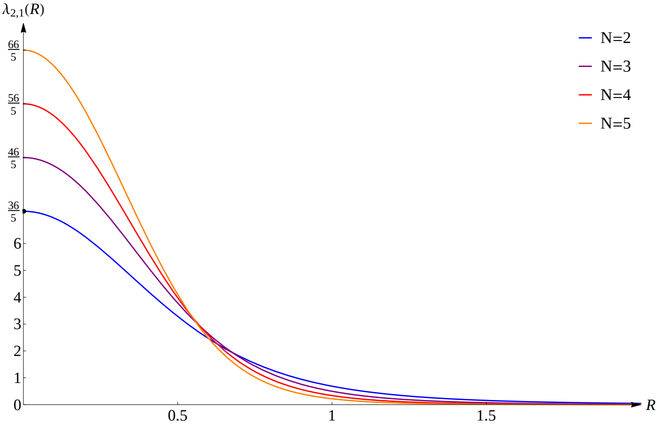

Figure: N=2,3,4,5

Further directions: Neumann problem, Poly-harmonic operators,...

Neumann problem for the Biharmonic operator

$$\begin{cases} \Delta^2 u = \lambda u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ -\mathrm{div}_{\partial\Omega}\left(D^2 u.\nu\right) - \frac{\partial(\Delta u)}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases}$$

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Neumann problem for the Biharmonic operator

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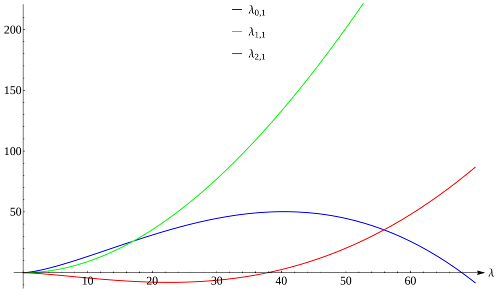

Figure: N=2

Further directions: Neumann problem, Poly-harmonic operators,...

- Neumann problem for $(-\Delta)^m$

$$\begin{cases} (-\Delta)^m u = \lambda u, & \text{in } \Omega, \\ N_1 u = N_2 u = \cdots = N_m u = 0, & \text{on } \partial\Omega, \end{cases}$$

$N_i u$ are the m natural boundary conditions, ordered according their order: N_1 is an operator of order m, N_2 is of order $m+1$,..., N_m is of order $2m-1$.

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- Steklov problem for $(-\Delta)^m$

$$\begin{cases} \Delta^m u = 0, & \text{in } \Omega, \\ N_1 u = N_2 u = \cdots = N_{m-1} u = 0, & \text{on } \partial\Omega, \\ N_m u = \lambda u, & \text{on } \partial\Omega, \end{cases}$$

with the same N_i

Further directions: mass concentration

- Behavior of $\lambda_j(\varepsilon)$ for mass concentration problem for the Biharmonic operator

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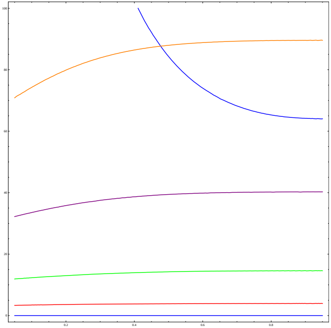

Figure: N=2, M=π, $\tau = 5$

- Behavior of $\lambda_j(\varepsilon)$ for mass concentration problem for the Biharmonic operator

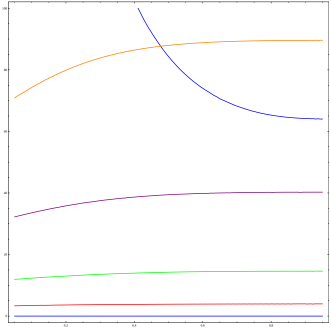

Figure: N=2, M=π, $\tau = 5$

- On the ball? On arbitrary Ω (also in the second order case)?

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Weak formulation

$$\int_{\Omega} D^2 u : D^2 \phi + \tau \nabla u \cdot \nabla \phi dx = \lambda \int_{\partial\Omega} u \phi d\sigma \,, \quad \forall \phi \in H^2(\Omega),$$

where $D^2 u : D^2 \phi = \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j}$

$$0 = \lambda_1[\Omega] < \lambda_2[\Omega] \le \cdots \le \lambda_j[\Omega] \le \cdots$$

Neumann vs Steklov, 2nd order example

For all $j \in \mathbb{N}$, $\lambda_j(\varepsilon) \to \lambda_j$ as $\varepsilon \to 0$

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For all $j \in \mathbb{N}$, $\lambda_j(\varepsilon) \to \lambda_j$ as $\varepsilon \to 0$

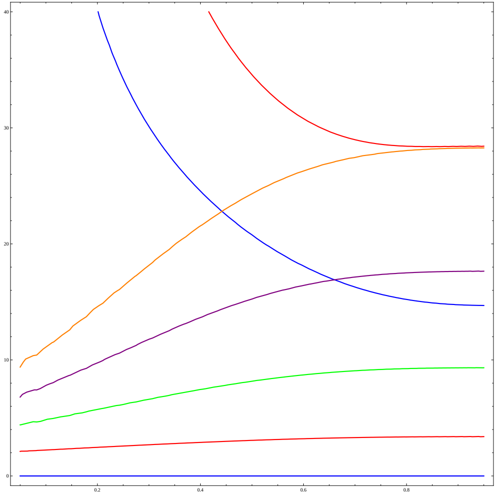

Figure: N=2, M=π

Isovolumetric perturbations

$$\mathcal{V}(\phi) = \int_{\phi(\Omega)} dy = \int_{\Omega} |\mathrm{det} D\phi| dx$$

Fix $\mathcal{V}_0 \in]0, +\infty[$

$$V(\mathcal{V}_0) = \{\phi \in \Phi[\Omega] \, : \, \mathcal{V}(\phi) = \mathcal{V}_0\}$$

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The function $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ if and only if

$$\sum_{l=1}^{|F|} \left(\lambda_F[\tilde{\phi}] \left(K v_l^2 + \frac{\partial v_l^2}{\partial \nu} \right) - \tau |\nabla v_l|^2 - |D^2 v_l|^2 \right) = c, \text{ a.e. on } \partial\tilde{\phi}(\Omega).$$

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Theorem (Buoso-P. 2014)

Let Ω be a domain of $\mathbb{R}^N$. Let $\tilde{\phi} \in \Phi(\Omega)$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of the problem in $\tilde{\phi}(\Omega)$, and let F be the set of $j \in \mathbb{N} \setminus \{0\}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$. Then $\Lambda_{F,s}$ has a critical point at $\tilde{\phi}$ on $V(\mathcal{V}(\tilde{\phi}))$, for all $s = 1, \dots, |F|$.

The fundamental tone

Lemma (Betta-Brock-Mercaldo-Posteraro 1999)

Let Ω be an open set in $\mathbb{R}^N$ and f be a continuous, non-negative, non-decreasing function defined on $[0,+\infty)$. Let us assume that the function

$$t \mapsto \big(f(t^{1/N}) - f(0)\big)t^{1-(1/N)}$$

is convex. Then

$$\int_{\partial\Omega} f(|x|)d\sigma \geq \int_{\partial\Omega^*} f(|x|)d\sigma,$$

where Ω^* is the ball centered at zero with the same volume as Ω.

The fundamental tone

Lemma (Hile-Xu 1993)

Let Ω be a bounded domain of class C^1 in $\mathbb{R}^N$. Then

$$\sum_{l=2}^{N+1} \frac{1}{\lambda_l(\Omega)} = \max\left\{ \sum_{l=2}^{N+1} \int_{\partial\Omega} v_l^2 d\sigma \right\},$$

where $\{v_l\}_{l=2}^{N+1}$ is a family in $H^2(\Omega)$ satisfying $\int_\Omega D^2 v_i : D^2 v_j + \tau \nabla v_i \cdot \nabla v_j dx = \delta_{ij}$ and $\int_{\partial\Omega} v_l d\sigma = 0$ for all $l = 2, ..., N+1$.